Math 122 / Problem Set 10

Written problems due Monday, December 5

Wednesday, November 30

1. A *prime ideal* P in a ring R is an ideal $P \neq R$ such that if $ab \in P$ then $a \in P$ or $b \in P$.

(a) Show that $R' = R/I$ is an integral domain if and only if I is a prime ideal.

(b) Show that if M is a maximal ideal then M is also a prime ideal.

Now let $\varphi : R \to R'$ be a ring homomorphism and let P' be a prime ideal of R'.

(c) Prove that $\varphi^{-1}(P') = \{r \in R : \varphi(r) \in P\}$ is a prime ideal of R.

(d) Give an example in which P' is a maximal ideal, but $\varphi^{-1}(P')$ is not maximal.

2. Prove that an integral domain with finitely many elements is a field. Is there an integral domain containing exactly 10 elements?

3. Let R be an integral domain.

(a) Prove that the polynomial ring $R[X]$ is an integral domain.

(b) Prove that the invertible elements of the polynomial ring $R[X]$ are the units in R.

4. Prove that the ring $\mathbb{F}_2[X]/(X^3+X+1)$ is a field, but that $\mathbb{F}_3[X]/(X^3+X+1)$ is not a field.

Reading: Artin §§11.1

Friday, December 2

5. Prove that there are infinitely many monic irreducible polynomials in $F[X]$ (F is a field). (*Hint:* You may want to recall Euclid's proof of the infinitude of primes.)

6. Factor the following polynomails into irreducible factors in $\mathbb{F}_p[X]$.

(a) $X^3 + X + 1$, $p = 2$

(b) $X^2 - 3X - 3$, $p = 5$

(c) $X^2 + 1$, $p = 7$

7. Prove that the greatest common divisor of two polynomials f and g in $\mathbb{Q}[x]$ is also their greatest common divisor in $\mathbb{C}[x]$.

Reading: Artin §11.2